Solution key VWO Mathematics B – Practice exam 1

Subject-specific marking rules and guidelines

1. For each error or mistake in calculation or notation a single point will be subtracted from the maximum score that can be obtained for that particular part of the question.

2. If a required explanation, deduction or calculation has been omitted or has been stated incorrectly 0 points will be awarded, unless otherwise stated in the solution key. This is also the case for answers obtained by the use of a graphic calculator. Answers obtained by the graphic calculator should indicate how the graphic calculator has been used to obtain the answer. Candidates must make sure they mention formulas applied or provide lists and calculation methods used in their answers.

3. A particular mistake in the answer to a particular exam question will lead to a deduction of points only once, unless the question is substantially simplified by the mistake and/or when the solution key specifies otherwise.

4. A repeated mistake made in the answer to different exam questions will lead to a deduction of points each time such a mistake has been made, unless the solution key specifies otherwise.

5. If only one example, reason, explication, explanation or any other type of answer is required and more than one has been given, only the first answer given will be graded. If more than one example, reason, explication, explanation or any other type of answer is required, only the first answers are graded, up to and including the number of answers specified by the exam question.

6. If the candidate fails to give a required unit in the answer to a question a single point will be subtracted from the total score, unless the unit has been specified in the exam question.

7. If during intermediate steps results are rounded, resulting in an answer different from one in which non-rounded intermediate results are used, one point will be subtracted from the total score. Rounded intermediate results may, however, be noted down.

Question 1.

a.	$f'(x) = [x^2]' \cdot \sqrt{x+1} + x^2 \cdot \left[\sqrt{x+1}\right]'$	1
	$f'(x) = 2x \cdot \sqrt{x+1} + x^2 \cdot \frac{1}{2\sqrt{x+1}}$	1
	$f'\left(-\frac{3}{4}\right) \left(= -\frac{3}{2} \cdot \sqrt{\frac{1}{4}} + \frac{9}{16} \cdot \frac{1}{2\sqrt{\frac{1}{4}}} \right) = -\frac{3}{4} + \frac{9}{16} = -\frac{3}{16}$ Note: when the candidate indicates that $f'\left(-\frac{3}{4}\right) = -\frac{3}{16}$ without showing the necessary intermediate steps, only one point should be awarded to this part of the question.	2
	Completing the proof	1
b.	$f(x) = 0$ gives $x = -1 \vee x = 0$	1
	The requested volume is given by: $$\pi \int_{-1}^{0} (f(x))^2 \, \mathrm{d}x$$	1
	Rewriting the expression $\pi \int_{-1}^{0} \left(x^2 \cdot \sqrt{x+1}\right)^2 \, \mathrm{d}x$ as $\pi \int_{-1}^{0} (x^5 + x^4) \, \mathrm{d}x$	2
	Determining an antiderivative of the integrand: $$\pi \int_{-1}^{0} (x^5 + x^4) \, \mathrm{d}x = \pi \left[\frac{1}{6}x^6 + \frac{1}{5}x^5\right]_{-1}^{0}$$	1
	$\pi \left[\frac{1}{6}x^6 + \frac{1}{5}x^5\right]_{-1}^{0} = \pi \left(0 - \left(\frac{1}{6} - \frac{1}{5}\right)\right) = \frac{1}{30}\pi$	1

Question 2.

a.	Noting that the vertical asymptote is given by: $x = 1$	1
	Rewriting the formula for $f(x)$ to $f(x) = x - 1 + \frac{1}{x-1}$	2
	Concluding from $f(x) = x - 1 + \frac{1}{x-1}$ that the slant asymptote is given by: $y = x - 1$	1
	Concluding from the formulas of the asymptotes that point P has coordinates $(1, 0)$.	1
b.	Simplifying $f(1+a)$ analytically to $$f(1+a) = \frac{a^2+1}{a} \left(= a + \frac{1}{a} \right)$$	2
	Simplifying $f(1-a)$ analytically to $$f(1-a) = \frac{a^2+1}{-a} \left(= -a - \frac{1}{a} \right)$$	2
	Concluding from the above that $$f(1+a) = -f(1-a)$$	1

Question 3.

a.	*Alternative 1:* $$f(-\ln(2)) = \frac{e^{-2\ln(2)}}{e^{-\ln(2)}+1} = \frac{\left(\frac{1}{4}\right)}{\left(\frac{3}{2}\right)}$$	2
	$$g(-\ln(2)) = \frac{2}{3} - e^{-\ln(2)} = \frac{2}{3} - \frac{1}{2}$$	1
	Show analytically that both expressions equal $\frac{1}{6}$ (And therefore that $f(-\ln(2)) = g(-\ln(2))$,and $x_A = -\ln(2)$)	1
	Alternative 2: Rewriting the expression given by $f(x) = g(x)$ to: $$2e^{2x} + \frac{1}{3}e^x - \frac{2}{3} = 0$$ (or something similar)	2
	Solving the found equation gives: $$e^x = -\frac{8}{12}\left(= -\frac{2}{3}\right) \ \vee \ e^x = \frac{6}{12}\ \left(= \frac{1}{2}\right)$$	1
	Concluding from the above that $x_A = \ln\left(\frac{1}{2}\right) = -\ln(2)$	1
b.	Determining the expression: $$F'(x) = e^x - \frac{1}{e^x+1} \cdot e^x$$ (Note: if the chain rule has not been applied or has not been applied correctly no points should be awarded to this part of the question)	2
	$$F'(x) = \frac{e^x(e^x+1)}{e^x+1} - \frac{e^x}{e^x+1}$$	1
	$$F'(x) = \frac{e^{2x} + e^x - e^x}{e^x+1} = \frac{e^{2x}}{e^x+1}\left(= f(x)\right)$$	1
c.	$g(x) = 0$ gives $x = \ln\left(\frac{2}{3}\right)$	1
	Noting that the area of V is given by: $$\int_{-\ln(2)}^{0} f(x)\,dx - \int_{-\ln(2)}^{\ln\left(\frac{2}{3}\right)} g(x)\,dx$$	1
	Evaluating the integral: $$\int_{-\ln(2)}^{0} f(x)\,dx = F(0) - F(-\ln(2)) \approx 0{,}2123$$	1
	Determining an antiderivative $G(x)$ of $g(x)$ to be $G(x) = \frac{2}{3}x - e^x$	1
	Evaluation of the integral: $$\int_{-\ln(2)}^{\ln\left(\frac{2}{3}\right)} g(x)\,dx = \left[\frac{2}{3}x - e^x\right]_{-\ln(2)}^{\ln\left(\frac{2}{3}\right)} = \left(\frac{2}{3}\ln\left(\frac{2}{3}\right) - e^{\ln\left(\frac{2}{3}\right)}\right) - \left(-\frac{2}{3}\ln(2) - e^{-\ln(2)}\right)$$ $$\approx 0{,}025$$	1
	Concluding that the area of V is equal to $0{,}187$ $\left(\text{or to } \frac{5}{3}\ln(3) - \frac{10}{3}\ln(2) + \frac{2}{3}\right)$	1

Question 4.

a.	*Alternative 1:* Determining the vectors: $$\overrightarrow{AP} = \begin{pmatrix}1\\3\end{pmatrix} \quad \text{and} \quad \overrightarrow{PB} = \begin{pmatrix}9\\-3\end{pmatrix}$$	2
	Calculating the dot product of the vectors $\overrightarrow{AP}$ and $\overrightarrow{PB}$: $$\overrightarrow{AP} \cdot \overrightarrow{PB} = 9 - 9 = 0$$	1
	So AP is perpendicular to BP (because the dot product is 0)	1
	Alternative 2: $$\text{slope}_{AP} = \frac{3}{1} = 3$$	1
	$$\text{slope}_{PB} = \frac{-3}{9} = -\frac{1}{3}$$	1
	$$\text{slope}_{AP} \cdot \text{slope}_{PB} = -1$$	1
	So AP is perpendicular to BP (because the product of the slopes is equal to -1)	1
	Alternative 3: $AP = \sqrt{1^2 + 3^2} = \sqrt{10}$	1
	$BP = \sqrt{9^2 + 3^2} = \sqrt{90}$	1
	$AB = 10$ so: $AP^2 + BP^2 = AB^2$	1
	So AP is perpendicular to BP (Pythagoras)	1
b.	The slope of the line through points B and C equals -2	1
	The line through points B and C is given by: $y = -2x + 20$	1
	Reducing the system $7y - x = 20 \wedge y = -2x + 20$ to $x = 8 \wedge y = 4$ (so $Q(8,4)$)	2
c.	The line through points A and Q is given by: $y = \frac{1}{2}x$ (or the line through points B and P is given by: $y = -\frac{1}{3}x + \frac{10}{3}$)	1
	$x_H\ (= x_C) = 4$ gives $y_H = 2$	1
	Noting that the radius of the inscribed circle is equal to the distance from point H to line ℓ	1
	$$d(H, \ell) = \frac{\lvert 7 \cdot 2 - 4 - 20 \rvert}{\sqrt{7^2 + 1^2}} = \frac{10}{\sqrt{50}} (= \sqrt{2})$$	1
	The inscribed circle of ΔPQR is given by: $$(x - 4)^2 + (y - 2)^2 = \left(\frac{10}{\sqrt{50}}\right)^2 \ (= 2)$$	1
	Rewriting the found equation to obtain: $$x^2 - 8x + y^2 - 4y + 18 = 0$$	1

Question 5.

a.	*Alternative 1:* Rewriting the equation $f(x) = g(x)$ as $\cos(x) = 2\cos(x)\sin(x) + 2\cos(x)$	1
	Reducing the equation $\cos(x) = 2\cos(x)\sin(x) + 2\cos(x)$ to: $\cos(x)\,(2\sin(x) + 1) = 0$	1
	The above equation gives: $\cos(x) = 0 \vee \sin(x) = -\frac{1}{2}$	1
	$\cos(x) = 0$ if $x = \frac{1}{2}\pi$ ($\vee\ x = \frac{3}{2}\pi$)	1
	$\sin(x) = -\frac{1}{2}$ if $x = \frac{7}{6}\pi\ \vee x = \frac{11}{6}\pi$	1
	So $x_B = \frac{7}{6}\pi$ and $x_c = \frac{11}{6}\pi$	1
	Alternative 2: Rewriting the equation $f(x) = g(x)$ as $\cos(x) = 2\cos(x)\sin(x) + 2\cos(x)$	1
	Reducing the equation $\cos(x) = 2\cos(x)\sin(x) + 2\cos(x)$ to $-\cos(x) = \sin(2x)$	1
	Rewriting the equation $-\cos(x) = \sin(2x)$ into the form $\cos(A) = \cos(B)$ or $\sin(A) = \sin(B)$. For example: $\cos(x - \pi) = \cos\left(\frac{1}{2}\pi - 2x\right)$ $\cos(x) = \cos\left(1\frac{1}{2}\pi - 2x\right)$ $\sin\left(1\frac{1}{2}\pi - x\right) = \sin(2x)$ $\sin\left(\frac{1}{2}\pi - x\right) = \sin(2x - \pi)$	2
	The solutions of the found equation are: $x = \frac{1}{2}\pi\ \vee\ x = \frac{7}{6}\pi$ ($\vee\ x = \frac{3}{2}\pi$) $\vee\ x = \frac{11}{6}\pi$	1
	So $x_B = \frac{7}{6}\pi$ and $x_c = \frac{11}{6}\pi$	1
b.	Noting that: $f'(x) = \dfrac{[\cos(x)]'(\sin(x) + 1) - [\sin(x) + 1]'\cos(x)}{(\sin(x) + 1)^2}$	1
	$f'(x) = \dfrac{-\sin(x)\,(\sin(x) + 1) - \cos(x)\cos(x)}{(\sin(x) + 1)^2}$	1
	$f'(x) = \dfrac{-\sin^2(x) - \sin(x) - \cos^2(x)}{(\sin(x) + 1)^2}$	1
	$f'(x) = \dfrac{-\sin(x) - 1}{(\sin(x) + 1)^2}$	1
	$f'(x) = \dfrac{-(\sin(x) + 1)}{(\sin(x) + 1)^2} = \dfrac{-1}{(\sin(x) + 1)}$	1
c.	Showing analytically that $f'\left(\frac{1}{2}\pi\right) = -\frac{1}{2}$	1
	The line tangent to the graph of f at point A intersects the y-axis at $y = \frac{\pi}{4}$ (so: $P\left(0, \frac{\pi}{4}\right)$)	1
	$AP = \sqrt{\left(\frac{\pi}{4}\right)^2 + \left(\frac{\pi}{2}\right)^2}$	1
	Simplifying $\sqrt{\left(\frac{\pi}{4}\right)^2 + \left(\frac{\pi}{2}\right)^2}$ to $\sqrt{\frac{5\pi^2}{16}} = \frac{\pi}{4}\sqrt{5}$	1

Question 6.

a.	*Alternative 1:* The velocity vector $\vec{v}(t)$ is given by: $$\vec{v}(t)=\begin{pmatrix}3t^2-3\\4t\end{pmatrix}$$	1
	Point P passes point $S(0,6)$ at $t=-\sqrt{3}$ and $t=\sqrt{3}$	1
	$$\vec{v}_1(-\sqrt{3})=\begin{pmatrix}6\\-4\sqrt{3}\end{pmatrix}$$ $$\vec{v}_2(\sqrt{3})=\begin{pmatrix}6\\4\sqrt{3}\end{pmatrix}$$	1
	$$\cos(\alpha)=\frac{\vec{v}_1\cdot\vec{v}_2}{\lvert\vec{v}_1\rvert\cdot\lvert\vec{v}_2\rvert}=\frac{-12}{84}\left(=-\frac{1}{7}\right)$$	2
	The requested angle equals $\cos^{-1}\left(-\frac{12}{84}\right)\approx 98{,}21°$	1
	Alternative 2: $$x'(t)=3t^2-3$$ $$y'(t)=4t$$	1
	Point P passes point $S(0,6)$ at $t=-\sqrt{3}$ and $t=\sqrt{3}$	1
	$$\frac{dy}{dx}=\frac{y'(\sqrt{3})}{x'(\sqrt{3})}=\frac{4\sqrt{3}}{6}\ (\text{or } 1{,}1547\ldots)$$ $$\frac{dy}{dx}=\frac{y'(-\sqrt{3})}{x'(-\sqrt{3})}=\frac{-4\sqrt{3}}{6}\ (\text{or } -1{,}1547\ldots)$$	1
	$$\tan^{-1}\left(\frac{4\sqrt{3}}{6}\right)=49{,}1\ldots°$$ $$\tan^{-1}\left(\frac{-4\sqrt{3}}{6}\right)=-49{,}1\ldots°$$	2
	The requested angle equals $49{,}1\ldots-(-49{,}1\ldots)\approx 98{,}21°$	1
b.	The acceleration vector $\vec{a}(t)$ is given by: $$\vec{a}(t)=\begin{pmatrix}6t\\4\end{pmatrix}$$	1
	Noting that the equation $\vec{v}(t)\cdot\vec{a}(t)=0$ has to be solved	1
	$$\vec{v}(t)\cdot\vec{a}(t)=18t^3-18t+16t=18t^3-2t$$	1
	Reducing the equation $\vec{v}(t)\cdot\vec{a}(t)=0$ to $t=0\vee t=-\frac{1}{3}\vee t=\frac{1}{3}$	2
	The requested points have coordinates $(0,0)$, $\left(\frac{26}{27},\frac{2}{9}\right)$ and $\left(-\frac{26}{27},\frac{2}{9}\right)$	2
c.	$x'(t)=0$ gives $t=-1\vee t=1$	1
	Noting that the requested distance is given by the difference between the x-coordinates of the tangent points	1
	$x(1)=-2$ and $x(-1)=2$	1
	The distance between the vertical tangent lines is equal to 4	1